Optimal Preconditioning for the Interval Parametric Gauss–Seidel Method

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Interval Linear Equations

$$Ax = b, \quad A \in \mathbf{A}, \; b \in \mathbf{b},$$

where

$$\mathbf{A} := [\underline{A}, \overline{A}] = [A^c - A^\Delta, A^c + A^\Delta],$$
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The Solution Set

$$\Sigma := \{x \in \mathbb{R}^n : \exists A \in \mathbf{A}, \exists b \in \mathbf{b} : Ax = b\}.$$

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Problem formulation

Find a tight interval vector enclosing Σ.

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The Interval Gauss–Seidel Method

$$\mathbf{z}_i := \frac{1}{(C\mathbf{A})_{ii}}\left((C\mathbf{b})_i - \sum_{j\neq i}(C\mathbf{A})_{ij}\mathbf{x}_j\right), \quad \mathbf{x}_i := \mathbf{x}_i \cap \mathbf{z}_i,$$

for $i = 1, \dots, n$.

Interval Parametric Systems

Interval Parametric System

$$A(p)x = b(p), \quad p \in \mathbf{p},$$

where

$$A(p) = \sum_{k=1}^{K} A^k p_k, \quad b(p) = \sum_{k=1}^{K} b^k p_k,$$

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Preconditioning and Relaxation

Relaxation to $Ax = b$, where

$$A \in \mathbf{A} := \sum_{k=1}^{K} (CA^k)\mathbf{p}_k, \quad b \in \mathbf{b} := \sum_{k=1}^{K} (Cb^k)\mathbf{p}_k.$$

The Parametric Interval Gauss–Seidel Method

$$\mathbf{z}_i := \frac{1}{\left(\sum_{k=1}^{K} (CA^k)_{ii}\mathbf{p}_k\right)} \left(\sum_{k=1}^{K} (Cb^k)_i \mathbf{p}_k - \sum_{j \neq i} \left(\sum_{k=1}^{K} (CA^k)_{ij} \mathbf{p}_k \right) \mathbf{x}_j \right),$$

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Optimal Preconditioner

Various criteria of optimality:

- minimize the resulting width, that is, the objective is $\min 2z_i^{\Delta}$,
- minimize the resulting upper bound, that is, the objective is $\min \overline{z}_i$,
- maximize the resulting lower bound, that is, the objective is $\max \underline{z}_i$.

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Minimal Width Preconditioner

Preliminaries

- For simplicity assume that $0 \in \mathbf{x}$ and $0 \in \mathbf{z}$
- Denote by c the ith row of C,
- Normalize c such that the denominator has the form of $[1, r]$ for some $r \geq 1$.

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Interval Gauss–Seidel Step

Then the operation of the ith step of the Interval Gauss–Seidel iteration is simplified to

$$\mathbf{z}_i := \sum_{k=1}^{K} (cb^k)\mathbf{p}_k - \sum_{j \neq i} \left(\sum_{k=1}^{K} (cA_{*j}^k)\mathbf{p}_k \right) \mathbf{x}_j$$

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The objective is $\min z_i^{\Delta}$.

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$$\text{minimize the width of} \quad \sum_{k=1}^{K}(cb^k)\mathbf{p}_k - \sum_{j\neq i}\left(\sum_{k=1}^{K}(cA_{*j}^k)\mathbf{p}_k\right)\mathbf{x}_j.$$

Denote

$$\begin{aligned}
\beta_k &:= |cb^k|, \quad k = 1, \dots, K, \\
\alpha_{jk} &:= |cA_{*j}^k|, \quad j = 1, \dots, n, \ k = 1, \dots, K, \\
\eta_j &:= \overline{\left(\sum_{k=1}^{K}(cA_{*j}^k)\mathbf{p}_k\right)\mathbf{x}_j}, \quad j \neq i, \\
\psi_j &:= \underline{\left(\sum_{k=1}^{K}(cA_{*j}^k)\mathbf{p}_k\right)\mathbf{x}_j}, \quad j \neq i.
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\eta_j \geq c \textstyle\sum_{k=1}^{K} A^k_{*j} p^c_k \underline{x}_j \pm \sum_{k=1}^{K} p^{\Delta}_k \underline{x}_j \alpha_{jk}, &\quad j \neq i, \\
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and the condition that the denominator is has the form of $[1, r]$

$$c\sum_{k=1}^K A_{*i}^k p_k^c - \sum_{k=1}^K p_k^{\Delta}\alpha_{ik} = 1.$$

Minimal Width Preconditioner

Optimization problem.

- Optimal preconditioner C found by n linear programming (LP) problems.
- each LP has $Kn + K + 3n - 2$ unknowns c, β_k, α_{jk}, η_j, and ψ_j, and $2Kn + 4n - 3$ constraints
- C needn't be calculated in a verified way.
- The problem is effectively solved in polynomial time.

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Practical Implementation

- Call the standard version using midpoint inverse preconditioner (or any other method),
- and after that tighten the enclosure by using an optimal preconditioner C.
- In our examples: one iteration with minimization of the upper bound, and one with maximization of the upper bound.

Example I

Example (Popova, 2002)

$$A(p) = \begin{pmatrix} 1 & p_1 \\ p_1 & p_2 \end{pmatrix}, \quad b(p) = \begin{pmatrix} p_3 \\ p_3 \end{pmatrix}, \quad p \in \mathbf{p} = ([0,1], -[1,4], [0,2])^T .$$

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Initial enclosure by the Parametric Interval Gauss–Seidel Method with midpoint inverse preconditioner:

- direct version: 7.66% of the width on average reduced
- residual form: 0% of the width on average reduced

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Initial enclosure as the interval hull of the relaxed system:

- direct version: 50% of the width on average reduced
- residual form: 12.56% of the width on average reduced

Example II

Example (Popova and Krämer, 2008)

$$A(p) = \begin{pmatrix} 30 & -10 & -10 & -10 & 0 \\ -10 & 10+p_1+p_2 & -p_1 & 0 & 0 \\ -10 & -p_1 & 15+p_1+p_3 & -5 & 0 \\ -10 & 0 & -5 & 15+p_4 & 0 \\ 0 & 0 & -5 & 5 & 1 \end{pmatrix}, \quad b(p) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where $p \in \mathbf{p} = [8, 12] \times [4, 8] \times [8, 12] \times [8, 12]$.

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Initial enclosure by the Parametric Interval Gauss–Seidel Method with midpoint inverse preconditioner:

- direct version: 15% of the width on average reduced
- residual form: 0% of the width on average reduced

Conclusion

Summary

- Optimal preconditioning matrix for the parametric interval Gauss–Seidel iterations.
- It can be computed effectively by linear programming.
- Preliminary results show that sometimes can reduce overestimation of the standard enclosures.

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Directions for Further Research

- Other types of optimality of preconditioners (S-preconditioners, pivoting preconditioners, etc.)
- Optimal preconditioners for other methods than the parametric interval Gauss–Seidel one.

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